Proof: **Existence:** $R_R = S_1^{n_1} \oplus \ldots \oplus S_k^{n_k}$ with simple S_i, $S_i \not\cong S_j$ for $i \neq j$ [C2.16].

$\Phi: R \xrightarrow{\simeq} End(R_R)$, $r \mapsto \Phi_r$ with $\Phi_r(x) = rx$ is a ring isomorphism

$\overset{[P2.14]}{\Longrightarrow}$ $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k)$ with $D_i = End(S_i)$.

Uniqueness: of simple modules: $S \in$ Mod-R simple

$\Rightarrow \exists$ epi $\varphi: S_1^{n_1} \oplus \ldots \oplus S_k^{n_k} \twoheadrightarrow S$.

$\Rightarrow S$ is a composition factor of $R_R \Rightarrow S \cong S_i$ for some i.

of matrix rings: Suppose also $R \cong M_{m_1}(E_1) \times \ldots \times M_{m_l}(E_l)$

w. E_i div. ring $\Rightarrow M_{m_i}(E_i) \cong V_i^{m_i}$ with V_i simple in Mod-$M_{m_i}(E_i)$.

$\Rightarrow V_1, \ldots, V_l$ are non-isomorphic simple Mod-R modules

[with $V_i M_{m_j}(E_j) = 0$ for $i \neq j$].

$\Rightarrow R_R \cong V_1^{m_1} \oplus \ldots \oplus V_l^{m_l}$.

Uniqueness of composition series implies (after renumbering),

$k = l$, $V_i \cong S_i$, $n_i = m_i$, $D_i = End(S_i) \cong End(V_i) \cong E_i$ $\square$

Remark: Analogous statement holds for left semisimple rings,

but now $D_i \cong End(S_i)^{op}$ because $End({}_RR) \cong R^{op}$.

Cor 2.19: R right semisimple $\Leftrightarrow$ R left semisimple

We just say: R is semisimple.

Def: A ring R is simple if $R \neq 0$ and it has no nonzero proper ideal. ideal = two-sided ideal

Exm: •) $M_n(D)$, D division ring (then $M_n(D)$ is also ordinion)

•) $A_1(k)$ if k is a field with char $k = 0$ (exercise)

So: R semisimple $\rightarrow$ $R = R_1 \times \_ \times R_k$ with R_i simple ordinion.

⚠ R simple ring $\not\Rightarrow$ R_R simple module

Prop 2.20 If R is a ring s.t.

$R = R_1 \times \_ \times R_k = R_1' \times \_ \times R_\ell'$ with simple rings,

then $k = \ell$, $R_i = R_i'$ after renumbering ($=$ not just $\cong$)

Proof: $R_i, R_j' \trianglelefteq R$ (ideal) and $R_i R = R_i$, $R_j' R = R_j'$

$\Rightarrow R_i = (R_i R_1') \times \_ \times (R_i R_\ell') \Rightarrow R_i R_j' \in \{0, R_i\}$ since

$R_i R_j' \trianglelefteq R_i$ and R_i is simple.

$0 \neq R_i \rightarrow \exists j: R_i R_j' = R_i$; also $R_i R_j' \trianglelefteq R_j' \rightarrow R_i R_j' = R_j'$

$\Rightarrow R_i = R_j'$ $\square$

2.5 Simple ordinion rings

Def: let $M \in \text{Mod-}R$.

(1) The annihilator of M is $\operatorname{ann}(M) := \{r \in R : mr = 0\}$

(Note: $\operatorname{ann}(M) \trianglelefteq R$)

(2) M is faithful if $\operatorname{ann}(M)=\underline{0}$, unfaithful otherwise.

(3) R is right primitive if it has a faithful simple right R-module

<u>Exm</u>: •) If R is commutative: R primitive $\Leftrightarrow$ R a field

[„$\Leftarrow$": $R=K$ field $\Rightarrow$ K_K simple faithful

„$\Rightarrow$": S_R simple faithful, $S_R \cong R/M$ M maximal ideal

$\rightarrow$ $M=\operatorname{ann}(R/M)=0$ $\Rightarrow$ $\underline{0}$ is maximal ideal $\Rightarrow$ R field]

•) Simple rings are right and left primitive: every simple M_R has $\operatorname{ann}(M_R)=\underline{0}$.

<u>Thm 2.21</u> TFAE for a ring R:

(a) R is simple ardinian

(b) R is simple and has a minimal nonzero right ideal

(c) R is right primitive and right ardinian.

(d) R is semisimple with a unique simple module up to isomorphism

(e) $R \cong M_n(D)$ with D a division ring, $n \geq 1$.

<u>Proof</u>: (a) $\Rightarrow$ (b) $R \neq \underline{0}$, so $\Omega = \{0 \neq I_R \leq R_R\} \neq \emptyset$. By ardinianity, Ω has a minimal element

(b) $\Rightarrow$ (c) Let I_R be a minimal nonzero right ideal. Then

$1 \notin \operatorname{ann}(I_R) \trianglelefteq R$, so $\operatorname{ann}(I_R)=\underline{0}$.

For every $r \in R$, $\varphi_r: I_R \to rI_R$, $x \mapsto rx$ has $\ker(\varphi_r) \in \{\underline{0}, I_R\}$, so $rI = 0$, or rI is simple. Thus $R_R = RI = \sum_{r \in R} rI_R$ is a sum of simple modules, hence semisimple, in particular artinian.

(c) $\Rightarrow$ (d) Let $M_R \in$ Mod-R be faithful simple.

Consider $\mathcal{F} := \{ f : R_R \to M_R^n : n \geq 0,\ f$ R-hom$\}$. Since R is right artinian $\{\ker f : f \in \mathcal{F}\}$ has a minimal element. Pick $f \in \mathcal{F}$ with $\ker f$ minimal.

<u>Claim</u>: $\ker f = \underline{0}$

[Suppose not. Let $0 \neq r \in \ker f$. $\mathrm{ann}(M) = \underline{0} \Rightarrow \exists m \in M : mr \neq 0$

Define $\tilde{f} : R \to M^n \oplus M$, $x \to (f(x), mr)$.

Then $\ker \tilde{f} \subsetneq \ker f$ $\lightning$]

So $R_R \hookrightarrow M_R^n$. M_R^n semisimple $\overset{T2.12}{\Longrightarrow} M_R^n = R_R \oplus K_R$

$\overset{C2.13}{\Longrightarrow} R_R$ semisimple, and $R_R \cong M_R^m$ for some $m \leq n$. by uniqueness (Remark after C2.13)

(d) $\Rightarrow$ (e) [T2.18]

(e) $\Rightarrow$ (a) ✓ $\square$

<u>Remark</u>: (1) $D \cong \mathrm{End}(V_R)$ with V_R the unique simple right R-module, but $D \cong \mathrm{End}({}_RW)^{op}$ with ${}_RW$ the unique simple left R-module.

(2) R simple right artinian $\Leftrightarrow$ R simple artinian $\Leftrightarrow$ R simple left artinian

2.6. Maschke's Theorem

Let $(G,\cdot)$ be a group. Fix a field K.

A representation of G is a group hom. $\rho: G \to GL(V)$ with V a K-vector space. If $\dim V = n < \infty$, $GL(V) \cong GL_n(K)$ (non-canonically, by choosing a basis). Representations are useful in studying groups (finite and infinite) ⇝ Representation Theory of Groups.

If $\rho: G \to GL(V)$, $\sigma: G \to GL(W)$ are representations, a homomorphism is a K-linear $T: V \to W$ s.t. $T(\rho(g)v) = \sigma(g)T(v)$ $\forall v \in V, g \in G$.

Representations of G form a category

Prop 2.22: Let G be a group, K a field. There is a category equivalence $\{\text{Representations of } G\} \longleftrightarrow \{(\text{left}) \ K[G]\text{-modules}\}$

Sketch: A $K[G]$-module structure on an abelian group $(M,+)$ corresponds to a ring hom. $\varphi: K[G] \longrightarrow \mathrm{End}(M_{\mathbb{Z}})$.

"$\leftarrow$": A $K[G]$-module M is a K-vector space $(\varphi|_K)$, and $\varphi|_G: (G,\cdot) \longrightarrow GL(M_K)$ is a group homomorphism.

"$\mapsto$": A repr. $\rho: G \to GL(V_K)$ gives rise to a monoid hom $\rho: (G,\cdot) \longmapsto (\underbrace{\mathrm{End}(V_K)}_{\text{ring}}, \circ)$. Using the UP of $K[G]$, this extends to a K-alg. hom $\tilde{\rho}: K[G] \longrightarrow \mathrm{End}(V_K)$ $\square$

Then irred. repr. $\hat{=}$ simple modules, completely reducible repr. $\hat{=}$ semisimple modules.

Thm 2.23 (Maschke's Theorem) If G is a finite group and K a field, then: $K[G]$ semisimple $\Leftrightarrow$ char $K \nmid |G|$

In particular: $\mathbb{C}[G]$ is semisimple.

Proof: "$\Leftarrow$" Let $I_{K[G]} \leq K[G]$ be a right ideal.

We show: the SES $0 \to I \hookrightarrow K[G] \xrightarrow{\pi} \underbrace{K[G]/I}_{=: V} \to 0$ of $K[G]$-modules split [T2.12(c)]. Since it splits as SES of K-vector spaces, $\exists \varphi : \operatorname{Hom}(V_K, K[G]_K)$ s.t. $\pi \circ \varphi = \mathrm{id}_V$.

char $K \nmid |G| \Rightarrow |G| \in K^\times$.

Define $\tilde{\varphi}(v) = \frac{1}{|G|} \sum_{g \in G} \varphi(vg) g^{-1}$

$\tilde{\varphi}$ is a $K[G]$-hom: K-linear ✓

$g' = hg \Rightarrow g = h^{-1} g' \Rightarrow g^{-1} = (g')^{-1} h$

$$\tilde{\varphi}(\underset{V}{v}\underset{G}{h}) = \frac{1}{|G|} \sum_{g \in G} \varphi(vhg) g^{-1} = \frac{1}{|G|} \sum_{g \in G} \varphi(vg) g^{-1} h = \tilde{\varphi}(v) h$$

$\pi \circ \tilde{\varphi} = \mathrm{id}$:
$$\pi \circ \tilde{\varphi}(v) = \frac{1}{|G|} \sum_{g \in G} \underbrace{(\pi \circ \varphi)(vg)}_{= vg} g^{-1} = v$$

"$\Rightarrow$": For $f = \sum_{g \in G} a_g g \in K[G]$, define $\varepsilon(f) = \sum_{g \in G} a_g$

(augmentation map), $I := \ker(\varepsilon)$ (augmentation ideal).

We show $I \cap J \neq 0$ for every nonzero right ideal $J \leq K[G]$.

Then $0 \to I \to K[G] \xrightarrow{\varepsilon} K \to 0$ is non-split, hence $K[G]$ not semisimple. Let $0 \neq x = \sum_{g \in G} a_g g \in J$

Case 1: $\varepsilon(x)=0 \Rightarrow x\in I\cap J$ ✓

Case 2: $\varepsilon(x)\neq 0$. Let $s := \sum'_{g\in G} g \Rightarrow \varepsilon(s) = |G|\cdot 1_K = 0$

$\Rightarrow s\in I$

$$\underset{\overset{\Uparrow}{I\cap J}}{xs} = \Big(\sum'_{g\in G} a_g g\Big)\Big(\sum'_{h\in G} h\Big) = \sum'_{g\in G} a_g \overbrace{\Big(\sum'_{h\in G} gh\Big)}^{=s} = \underset{\overset{\Uparrow}{K^\times}}{\varepsilon(x)}\, s \neq 0$$

□

Prop 2.24 If $|G|=\infty$, then $K[G]$ is not semisimple

Proof: Consider again the augmentation map ε, augmentation ideal I.

Suppose $K[G]$ is semisimple.

$\Rightarrow 0 \to I \hookrightarrow K[G] \xrightarrow{\varepsilon} K \to 0$ splits.

Let $\varphi: K \to K[G]$ s.t. $\varepsilon\circ\varphi = id_K$, φ $K[G]$-hom.

$\Rightarrow 0 \neq \varphi(1) =: f = \sum'_{g\in G} a_g g$ with finitely many $a_g\neq 0$ (but at least one!)

Let $h\in G$ with $a_h\neq 0$.

$K[G]$-module structure on K: $\forall g\in G\ \exists \lambda_g\in K$: $1_K\cdot g = \lambda_g$

$$\left.\begin{array}{ll} \Rightarrow & \varphi(g) = \varphi(\lambda_g) = \varphi(1)\lambda_g = f\lambda_g \\ \text{o.t.o.h} & \varphi(g) = \varphi(1_K)g = fg \neq 0 \end{array}\right\} \Rightarrow fg = f\lambda_g \neq 0$$

$\Rightarrow \forall g\in G$: hg appears in support of f ↯ only finitely many $a_g\neq 0$. □

3. Jacobson Radical

Let $M \in \text{Mod-}R$, $X \subseteq M$

The annihilator of X, $\operatorname{ann}(X) := \{ r \in R : \forall x \in X : xr = 0 \}$ is a right ideal. $\operatorname{ann}(\underset{M}{\overset{\in}{m}}) := \operatorname{ann}(\{m\})$, $\operatorname{ann}(X) = \bigcap_{x \in X} \operatorname{ann}(x)$

If $X \leq_R M_R$, then $\operatorname{ann}(X) \overset{\text{ideal.}}{\trianglelefteq} R$.

<u>Def:</u> $I \trianglelefteq R$ is right primitive if there is a simple $M \in \text{Mod-}R$ s.t. $I = \operatorname{ann}(M)$ [$\Leftrightarrow$ R/I is right primitive.]

<u>Exm:</u> ·) $I \trianglelefteq R$ maximal $\Rightarrow$ I primitive

[By Zorn's Lemma, I is contained in a max. right ideal J

$\rightarrow$ $(R/J)_R$ simple, $\operatorname{ann}(R/J) \supseteq I$ and $\operatorname{ann}(R/J) = I$ by maximality of I]

·) If R commutative: I primitive $\Rightarrow$ I maximal

[$I = \operatorname{ann}(R/J)$, $J \leq_R R_R$ maximal, J two-sided $\Rightarrow$ $J = \operatorname{ann}(R/J) = I$.]

<u>Def:</u> The Jacobson radical is $J(R) := \bigcap_{\substack{I \trianglelefteq R \\ I \text{ right primitive}}} I = \bigcap_{\substack{M \in \text{Mod-}R \\ M \text{ simple}}} \operatorname{ann}(M_R)$

$\mathcal{J}(R)$ = "rad R"

<u>Note:</u> $J(R) \trianglelefteq R$ and $J(R) \subsetneq R$ unless $R = 0$

[<u>Properness:</u> If $R \neq 0$, there exists a maximal right ideal $I \neq R$ (Zorn), $\operatorname{ann}(R/I) \subsetneq R$.]

Lemma 3.1: $J(R) \overset{A}{=} \overset{B}{\bigcap}\{J : J_R \leq R_R \text{ maximal}\}$

$\overset{C}{=} \{r \in R : \forall x \in R : 1 - rx \text{ right invertible}\}$

$\overset{D}{=} \{r \in R : \forall x, y \in R : 1 - xry \text{ invertible}\}$

Note: Since the final condition is left/right symmetric, we also get the corresponding statements on the left! E.g., $J(R) = \bigcap_{\substack{I \trianglelefteq R \\ I \text{ left primitive}}} I$.

Proof: „$A \subseteq B$": Let $J_R \leq R_R$ be maximal, $r \in J(R)$

$\Rightarrow (R/J)_R$ simple $\Rightarrow (R/J)r = 0 \Rightarrow Rr \subseteq J \Rightarrow r \in J$.

„$B \subseteq C$": $\forall$ maximal $J \leq R_R$, $rx \in J \Rightarrow 1 - rx \notin J$ (otherwise $1 \in J$)

So $(1-rx)R$ is not contained in a max. right ideal

$\Rightarrow (1-rx)R = R \Rightarrow \exists y \in R, \ (1-rx)y = 1$.

„$C \subseteq A$": Suppose $Mr \neq 0$ for some simple M

$\Rightarrow \exists m \in M: \ mr \neq 0 \xrightarrow{\text{simplicity}} mrR = M \Rightarrow \exists x \in R: \ m = mrx$

$\Rightarrow m(1-rx) = 0 \xrightarrow{1-rx \text{ right inv.}} m = 0$ ↯.

„$C = D$": $D \subseteq C$ ✓ $C \subseteq D$: Let $r \in J(R)$, $x, y \in R$

Show: $1 - xry \in R^\times$

$xr \in J(R) \xrightarrow{C=A} \exists z \in R: \quad (1-xry)z = 1 \Rightarrow z$ left inv.

$z = 1 - \underbrace{(-xryz)}_{\in J(R)} \xrightarrow{C=A} z$ right invertible

$\Rightarrow z \in R^\times$

$\Rightarrow 1 - xry \in R^\times$ □

<u>Cor 3.2</u>: $J(R)$ is the largest right ideal I s.t. $1+I \subseteq R^\times$.

<u>Exm</u> 1) $J(\mathbb{Z}) = 0$, $\quad J(\mathbb{Z}/p^n\mathbb{Z}) = p\mathbb{Z}/p^n\mathbb{Z}$

2) If $\prod_{i\in I} R_i$ is a product of rings, $J\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} J(R_i)$.

3) R simple $\Rightarrow J(R) = 0$

4) R semisimple $\Rightarrow J(R) = 0$ (e.g. by 2), 3) + Wedderburn-Artin)

5) $J(M_n(R)) = M_n(J(R))$ [Exercise]